

0-brane Quantum Chemistry

VÁCLAV KAREŠ¹

*Institute for Theoretical Physics, Masaryk University,
Kotlářská 2, 611 37, Brno, Czech republic*

Abstract

We apply two different numerical methods to solve for the boundstate of two 0-branes in three dimensions. One method is developed by us in this work and we compare it to a method existing in the literature. In spite of considering only three dimensional Minkowski space we obtain interesting results which should give some basic understanding of the behaviour of 0-branes.

1 Introduction

D-branes are very important non-perturbative objects in string theory. Their existence is essential since they are needed, among other things, for various string theory dualities to work. The most primitive definition of a D-brane is as a hyper surface on which open strings end.

In this work we are interested only in D0-branes. They play a particularly important role since they are the basic constituents of Matrix theory [1], which, being a suggestion for a non-perturbative definition of string theory (or M-theory [2]), deserves particular attention. It is also possible to think about them as bound states of higher dimensional unstable D-branes [3, 4]. Furthermore, as was shown by Myers [5], they can also form bound states with all the properties of higher dimensional branes. It is therefore clear that D0-branes have many interesting properties that make them worth studying in detail. Concretely this means studying supersymmetric quantum mechanics which is an interesting topic in itself. Although this topic has been studied before [6]-[14] we feel that there are still unresolved issues. In particular one could use a computer to compute properties of D0-branes which are not possible to address analytically because of the complexity of the theory. This is the goal of this work, to develop a “0-brane Quantum Chemistry”. It should be mentioned that similar issues have

¹*E-mail: kares@physics.muni.cz*

been addressed in [6]. However, we use a different method which we compare to the method developed in [6] (to which we also suggest certain improvements).

More concretely, in this work we try to find the bound state of two D0-branes in three dimensional Minkowski space (which is really a toy-model for the real situation, D0-branes in ten dimensional Minkowski space). It should also be noticed that the really interesting cases where the D0-brane theory is thought to describe macroscopic supergravity states is achieved by taking the number of zero branes to infinity.

2 The model

The low energy physics of N parallel D p -branes is governed by the dimensional reduction of $9 + 1$ dimensional $\mathcal{N} = 1$ supersymmetric Yang-Mills theory with $U(N)$ gauge group to $p+1$ dimension [15]. The center of mass motion is governed by the overall $U(1)$ factor so if we are interested in relative motion only, we can choose the gauge group to be $SU(N)$. In the case we are interested in, the relative motion of two 0-branes, we thus choose the gauge group to be $SU(2)$. The action is [7]

$$S = \int dt \left[\frac{1}{2g_s} \dot{X}_i^a \dot{X}_i^a + \frac{i}{2} \psi_A^a \dot{\psi}_A^a - \frac{1}{4g_s} (\epsilon^{abc} X_i^b X_j^c)^2 + \frac{i}{2} \epsilon^{abc} X_i^a \psi_A^b (\gamma^i)_{AB} \psi_B^c \right. \\ \left. + \frac{1}{g_s} \epsilon^{abc} \dot{X}_i^a A_0^b X_i^c + \frac{1}{2g_s} (\epsilon^{abc} A_0^b X_i^c)^2 - \frac{i}{2} \epsilon^{abc} A_0^a \psi_A^b \psi_A^c \right] \quad (1)$$

and the Hamiltonian derived from this is [7]

$$H = \frac{g_s}{2} (\pi_i^a)^2 + \frac{1}{4g_s} (\epsilon^{abc} X_i^b X_j^c)^2 - \sum_{i=1}^7 \epsilon^{abc} X_i^a \bar{\chi}_A^b (\tilde{\gamma}^i)_{AB} \chi_B^c \\ - \frac{1}{2} \epsilon^{abc} X_8^a (\chi_A^b \chi_A^c - \bar{\chi}_A^b \bar{\chi}_A^c) - \frac{i}{2} \epsilon^{abc} X_9^a (\chi_A^b \chi_A^c + \bar{\chi}_A^b \bar{\chi}_A^c) \quad (2)$$

together with the constraint one gets from varying (1) with respect to A_0

$$G^a \equiv \epsilon^{abc} (X_i^b \pi_i^c - i \bar{\chi}_A^b \chi_A^c) = 0. \quad (3)$$

On quantum level, we restrict our Hilbert space to vectors which satisfy

$$G^a |\Psi\rangle = 0 \quad (4)$$

that is, our physical space is gauge invariant because G^a are gauge generators.

We will only study motion of two 0-branes in three dimensional Minkowski space. We hope that this gives us the basic behavior of 0-branes and also an understanding of the full problem. This problem of two branes is also described in [7] but we study it in a different way. In the three dimensional case the action

is given by dimensional reduction of $\mathcal{N} = 1$ supersymmetric Yang-Mills theory with $SU(2)$ gauge group in $2 + 1$ dimension to $0 + 1$ dimension. The Hamiltonian takes the slightly simpler form

$$H = \frac{g_s}{2} (\pi_i^a)^2 + \frac{1}{4g_s} (\epsilon^{abc} X_1^b X_2^c)^2 - \frac{1}{2} \epsilon^{abc} X_1^a (\chi^b \chi^c - \bar{\chi}^b \bar{\chi}^c) - \frac{i}{2} \epsilon^{abc} X_2^a (\chi^b \chi^c + \bar{\chi}^b \bar{\chi}^c) \quad (5)$$

where X_i are fields in the $SU(2)$ adjoint representation and χ is a complex fermion also in the adjoint representation. Of course we still have to impose gauge invariance (4). In fact, the gauge invariance complicates things somewhat since we would like to separate out gauge invariant degrees of freedom from pure gauge degrees of freedom in our basic quantum mechanical operators X_i^a and π_i^a . Let us focus on physical content of the X_i^a . It contains six components (the gauge index runs over three values and the space index i runs from 1 to 2). We know that we can remove three of these variables using gauge transformations so only three variables are observable. These three variables should describe the relative position of two pointlike objects in two space dimensions. We draw the conclusion that one of the physical variables do not have the interpretation of a coordinate but rather as some internal auxilliary degree of freedom.

To get some further insight into this problem it is necessary to investigate the bosonic vacuum of the theory. It is possible to explicitly separate the gauge degrees of freedom from X_i^a by decomposition it in matrix form [12]

$$(X)_{ai} = (\psi)_{ar} (\Lambda)_{rs} (\eta)_{si} . \quad (6)$$

Here the matrix ψ is an group element in the adjoint representation of $SU(2)$. Thus when the gauge group acts on X_i^a , ψ just changes by ordinary gauge group multiplication (from the left). We will parametrise the group element ψ by the "angles" α, β, γ (A.5). This decomposition has the advantage that all the gauge dependence sits in ψ and all the other matrices are gauge invariant.

In an analogous way we have separated out the dependence on rotations in space. Namely, performing an $SO(2)$ rotation in space we have an element of $SO(2)$ acting from the right on the matrix X_{ai} . Thus we can separate out the dependence on the angle in space (we will call it ϕ) by saying that η is a group element of $SO(2)$.

We are left with the matrix Λ (A.2) which by construction is both gauge and space rotation invariant

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \\ 0 & 0 \end{pmatrix} . \quad (7)$$

The bosonic potential in (5) is gauge and rotation invariant and in the new decomposition coordinates depends only on two λ_i which have length dimension (Fig. 1).

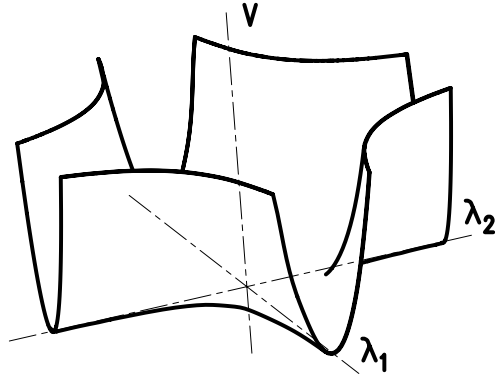


Figure 1: Bosonic potential

The parametrisation

$$\lambda_1 = r \cos \theta; \lambda_2 = r \sin \theta \quad (8)$$

is the only way how to obtain exactly one variable, r , with the dimension length, which could represent relative distance of two branes. The dimensionless θ is the auxilliary coordinate. The potential in this coordinate reads

$$\frac{1}{8g_s} r^4 \sin^2 2\theta. \quad (9)$$

Looking at the picture we can draw some interesting conclusions. If we fix a point on the bosonic vacuum (a classical static configuration with minimum energy), that is on the axes, we can study the behavior of the potential for small fluctuations of the auxilliary variable θ . We see that for large r the θ fluctuation are very much suppressed but at small r , θ will be allowed to fluctuate. This can be interpreted to mean that when the branes get close to each other, they can start to move also in the θ direction. Thus, θ is an auxilliary coordinate which is visible only when the branes come close together.

The above discussion included only the bosonic degrees of freedom, we should keep in mind that the fermionic degrees of freedom can (and will) change this behavior somewhat. In essence, the Pauli repulsion will try to spread out the wavefunction as much as possible.

When we use the standard operator representation of π_i^a and X_i^a the first term in the Hamiltonian (5) is proportional to the Laplacian which we have to rewrite in the decomposition coordinates above to separate out gauge using formula

$$\frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j) \quad (10)$$

where the g is the metric which is trivial in X_i^a coordinates. It is also good idea

to rewrite the Laplacian in terms of gauge angular momenta $L^a \equiv \epsilon^{abc} X_i^b \pi_i^c$:

$$\begin{aligned} L^1 &= -i \cot \gamma \sin \alpha \frac{\partial}{\partial \alpha} + i \csc \gamma \sin \alpha \frac{\partial}{\partial \beta} + i \cos \alpha \frac{\partial}{\partial \gamma} \\ L^2 &= -i \cot \gamma \cos \alpha \frac{\partial}{\partial \alpha} + i \csc \gamma \cos \alpha \frac{\partial}{\partial \beta} - i \sin \alpha \frac{\partial}{\partial \gamma} \\ L^3 &= i \frac{\partial}{\partial \alpha} , \end{aligned} \quad (11)$$

and physical angular momentum

$$L^0 \equiv \epsilon^{ij} X_i^a \pi_j^a = -i \frac{\partial}{\partial \phi} \quad (12)$$

since their action on states with given gauge and rotational properties is simple. In particular, the (bosonic) ground state should have total spin equal to zero and be gauge invariant. All angular momenta are Killing vectors of the metric.

In Appendix B we derive the lagrangian expressed in terms of gauge invariant variables and angular momenta

$$L = \frac{1}{r^5} \frac{\partial}{\partial r} r^5 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin 4\theta} \frac{\partial}{\partial \theta} \sin 4\theta \frac{\partial}{\partial \theta} - (\Psi \Pi^{-1} \Psi^{-1})_{\mu\nu} L^\mu L^\nu . \quad (13)$$

Here L^μ $\mu = 0, 1, 2, 3$ is a compact notation for the (physical and gauge) angular momenta defined above. Furthermore, we have defined

$$\Psi = \begin{pmatrix} -1 & \\ & \psi \end{pmatrix} \quad (14)$$

and Π

$$\begin{pmatrix} r^2 & & & r^2 \sin 2\theta \\ & r^2 \sin^2 \theta & & \\ & & r^2 \cos^2 \theta & \\ r^2 \sin 2\theta & & & r^2 \end{pmatrix} . \quad (15)$$

To find the (bosonic) ground state we will find all gauge invariant states with spin zero. The first state is the vacuum state $|0\rangle$. Then we may act with the fermionic creation operators χ^a on the vacuum to find new states. The following states has total spin zero and they are gauge invariant

$$|r\rangle = \frac{1}{2} \psi_{ar} |a\rangle = \frac{1}{2} e^{i\phi} \psi_{ar} \epsilon^{abc} \chi^b \chi^c |0\rangle . \quad (16)$$

That is, they satisfy

$$G^a |r\rangle = 0 . \quad (17)$$

The most general gauge invariant wavefunction with total spin zero can then be written

$$g(r, \theta) |0\rangle + f_r(r, \theta) |r\rangle . \quad (18)$$

It would also be possible to construct the superpartner ground state with the help of

$$\begin{aligned} |r'\rangle &= e^{i\phi} \psi_{ar} \chi^a |0\rangle \\ |0'\rangle &= \frac{1}{6} e^{2i\phi} \epsilon^{abc} \chi^a \chi^b \chi^c |0\rangle . \end{aligned} \quad (19)$$

Now we study what happens when we act with the Laplacian (13) on this wavefunction and using the result we write the Hamiltonian matrix in this base. Let us start with second part of the Laplacian which is the relevant bosonic piece

$$\begin{aligned} -(\Psi \mathbf{\Pi}^{-1} \Psi^{-1})_{ab} L^a L^b |r\rangle &= |s\rangle \mathbf{\Pi}_{sr}^{-1} - |r\rangle \text{Tr} \mathbf{\Pi}^{-1} \\ -2(\Psi \mathbf{\Pi}^{-1} \Psi^{-1})_{0a} L L^a |r\rangle &= 2i |u\rangle \mathbf{\Pi}_{0s}^{-1} \epsilon_{usr} \\ -(\Psi \mathbf{\Pi}^{-1} \Psi^{-1})_{00} L L |r\rangle &= -|r\rangle \mathbf{\Pi}_{00}^{-1} \end{aligned} \quad (20)$$

where Tr is only on the gauge indexes. Finally the fermionic interaction in the Hamiltonian gives us

$$\begin{aligned} H_F |r\rangle &= -|0\rangle (\Lambda\eta)_{r1} + i|0\rangle (\Lambda\eta)_{r2} \\ H_F |0\rangle &= -|r\rangle (\Lambda\eta)_{r1} - i|r\rangle (\Lambda\eta)_{r2} . \end{aligned} \quad (21)$$

So the full Hamiltonian matrix elements are

$$\begin{aligned} H_{rs} &= -\frac{g_s}{2} \left(h + \mathbf{\Pi}_{rs}^{-1} - \delta_{rs} \text{Tr} \mathbf{\Pi}^{-1} + 2i \mathbf{\Pi}_{0u}^{-1} \epsilon^{rus} - \delta_{rs} \mathbf{\Pi}_{00}^{-1} \right) + \frac{1}{8g_s} r^4 \sin^2 2\theta \\ H_{0r} &= -(\Lambda\eta)_{r1} + i(\Lambda\eta)_{r2} \\ H_{00} &= -\frac{g_s}{2} h + \frac{1}{8g_s} r^4 \sin^2 2\theta \end{aligned} \quad (22)$$

where

$$h = \frac{1}{r^5} \frac{\partial}{\partial r} r^5 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin 4\theta} \frac{\partial}{\partial \theta} \sin 4\theta \frac{\partial}{\partial \theta} . \quad (23)$$

Notice that the Hamiltonian we have obtained is the same as in [14] but differs from the one used in [7].

3 Numerical calculation I

This section gives an overview of solving the Hamiltonian eigenvalue problem (5) using the numerical renormalized Numerov method [16]. With it one can solve for the discrete spectra of a one dimensional operator of the form

$$-\frac{1}{2} \frac{\partial^2}{\partial x^2} \mathbf{1} + \mathbf{V}(x) \quad (24)$$

which acts on $L[a, b] \otimes \mathbf{C}^n$ with Dirichlet boundary conditions. However, our Hamiltonian depends on the *two* coordinates r and θ (the other coordinates, the angles, are fixed by the requirement that we are studying only gauge invariant states with spin zero). The “kinetic” term in our Hamiltonian (i.e. the term which contains the derivatives) is

$$\frac{1}{r^5} \frac{\partial}{\partial r} r^5 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin 4\theta} \frac{\partial}{\partial \theta} \sin 4\theta \frac{\partial}{\partial \theta} \quad (25)$$

and we see that it is naively not of the form required above. Let us sketch briefly how to modify our problem to be able to apply the method. An arbitrary wavefunction can be written in the form

$$|\Psi\rangle = \sum_{i,j} \Psi_{ij}(r) Y_{ij}(\theta) e_i \quad (26)$$

where $\{Y_{ij}(\theta), j = 1, \dots\}$ is a complete basis of functions (with appropriate boundary conditions) in θ (the explicit choice of basis does not have to be the same for different values of the index i but can be chosen to optimize the numerics). The functions $\{\Psi_{ij}, j = 1, \dots\}$ (which we will compute by the Numerov method) one can think of as being “combination coefficients” depending continuously on r . $\{e_i\}$ is the standard base spanning \mathbf{C}^n , in our case it is the four dimensional complex vector space on which the Hamiltonian matrix acts. Choosing a concrete basis depends only on the Hamiltonian domain which depends on one of the selfconjugated extensions of the Hilbert space (an extension of the Hilbert space such that the Hamiltonian operator is hermitian) but we will rather apply a physical principle which will be described later. The expectation values of the Hamiltonian in the base

$$Y_{ij} e_i \text{ (no sum)} \quad (27)$$

gives the same number of coupled equations for the radial part as the number of Ψ_{ij} in (26). Thus the problem is now correctly defined and the matrix representation of the expectation value of the Hamiltonian in the basis above forms the potential which is used in the one dimensional Hamiltonian (24). To get 1 in the kinetic term we have to orthonormalize the base. Furthermore, to be able to use the Numerov method on a computer we need a finite basis which means that we need to “cut off” or restrict the base to be finite. The rescaling of the wavefunction $|\Psi\rangle$ by the factor $r^{5/2}$ transforms the radial part of (25) to the operator

$$\frac{\partial^2}{\partial r^2} - \frac{15}{4} r^2 \quad (28)$$

and changes boundary condition to Dirichlet.

Let us now describe the physical principle we use to choose the basis functions $Y_{ij}(\theta)$ for our Hamiltonian (5).

We define the wavefunction on the interval $\theta \in [0, \pi/4]$ (A.27). It is not necessary to choose this particular interval, one could, for instance, select the interval $[-\pi/4, 0]$ instead of the above mentioned. Using the identification $\tilde{\theta} = -\theta$ the Hamiltonian defined on the interval $\tilde{\theta} \in [-\pi/4, 0]$ acting on states with total spin zero is connected to our original Hamiltonian on the interval $\theta \in [0, \pi/4]$ by the unitary transformation

$$H(\tilde{\theta}) = U^\dagger H(\theta) U \quad (29)$$

where

$$U = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}. \quad (30)$$

It is also possible to consider other interval $\tilde{\theta} \in [\pi/4, \pi/2]$. Using the identification $\tilde{\theta} = \pi/2 - \theta$ the corresponding Hamiltonian can be also obtained by a unitary transformation with the matrix

$$U = \begin{pmatrix} 1 & & & \\ & -i & & \\ & i & & \\ & & & 1 \end{pmatrix}. \quad (31)$$

The wavefunctions of course also transform under the unitary transformation

$$|\Psi(\theta)\rangle = U|\Psi(\tilde{\theta})\rangle. \quad (32)$$

If we require that the wavefunctions be everywhere smooth, the above condition severely restricts the possible wavefunctions and in particular the basis wavefunctions that we can use. We note that this principle was not used in [7] and hence the θ derivative of their groundstate wavefunction at $\theta = \pi/4$ is not well defined.

For the bosonic part of the wavefunction we will use the basis states

$$\cos(4j\theta)e_1 \quad j = 0, \dots \quad (33)$$

and for the fermionic part we will use

$$-i \cos \theta \cos(2j\theta)e_2 + (-1)^j \sin \theta \cos(2j\theta)e_3 \quad j = 0, \dots \quad (34)$$

They nicely cancel the divergences in the Hamiltonian coming from the Laplacian. Of course these basis functions satisfy the boundary condition above and they also form a complete basis for the functions with the given boundary conditions.

However, the basis is not orthonormal due to the non trivial θ part in the measure (A.26). So we need to orthonormalize these basis functions to get **1** in (24). Doing so, using the Gramm-Schmidt procedure, we diagonalize the θ part of the Laplacian. Notice that these two sets of basis functions also correspond to the bases which are implicitly used in the Wosiek method [6] when one are doing calculations with states with total spin zero (the basis functions above depend on the total spin). This will be shown in section 5. Thus we will be able to compare our results and our method very directly with the results obtained by the Wosiek method [6].

It is not obvious how much our results for a fixed number of basis functions fit the exact solutions which one would get using the complete basis. To get some intuition for how the general solution would look like we will repeat the calculations increasing the number of basis functions each time and hopefully one can extrapolate the result to the exact case. At least we should be able to make an intelligent guess at the properties of the exact solution.

To study the groundstate of our Hamiltonian (which, because of supersymmetry, should have energy zero [11]) is a good test for the method described above.

There are some results for coupling constant $g_s = 0.1$ in the table 1 (p. 17) where N is the number of the test functions (33,34) and E is their corresponding groundstate energy. The dependence of the energy on the number of functions we include in the basis given in the last two columns of the table is as follows

$$E = \frac{1.44}{N} . \quad (35)$$

As was claimed in [6], one can therefore predict the groundstate energy as a function of the number of basis functions with very high accuracy. We therefore see that in any concrete numerical calculation (using a finite basis) we do not expect to get zero energy. Only in the (numerically unobtainable) case of infinite basis do we get zero energy.

The picture (Fig. 2) is the probability density for the case with the highest number of basis functions in the table. The following two pictures show the bosonic contribution coming from (33) and the fermionic one coming from (34).

The domain of these plots is $(r, \theta) \in [0, 2.2] \times [-\pi/4, \pi/4]$. The hill of the probability density is located at the boson potential valley (9) and isolines represents sections for fixed r , θ and one for fixed density on each picture. The maximum of the probability density of any constant r section is in the potential valley ($\theta = 0$). Notice that the global maximum is not at $r = 0$. This is probably an effect of the fermion Pauli repulsion which can be seen from the purely fermionic contribution to the probability density which is zero at $r = 0$.

Increasing the number of basis functions, the only thing that happens is that the global maximum moves slowly to larger and larger r at the same time as the whole wavefunction becomes more spread out in r but more peaked in θ . One

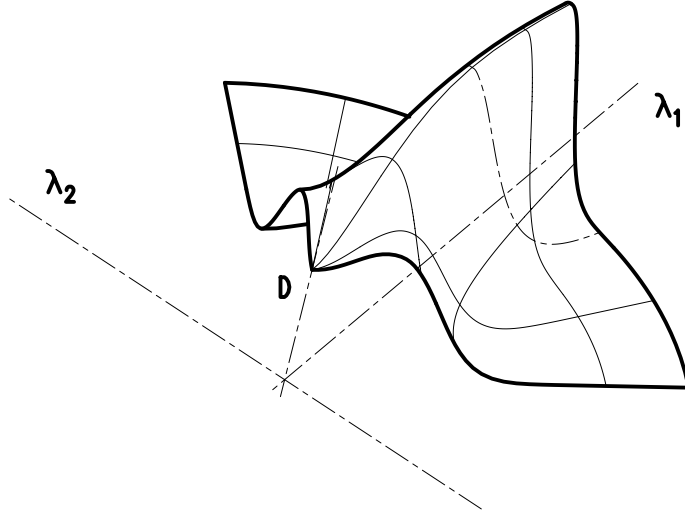


Figure 2: Probability density

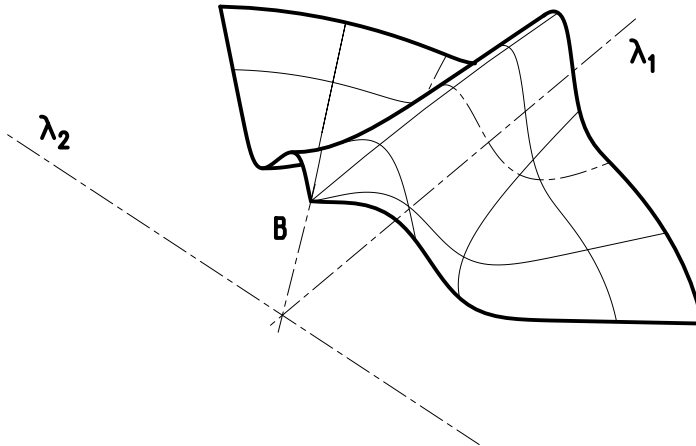


Figure 3: Bosonic part of the probability density

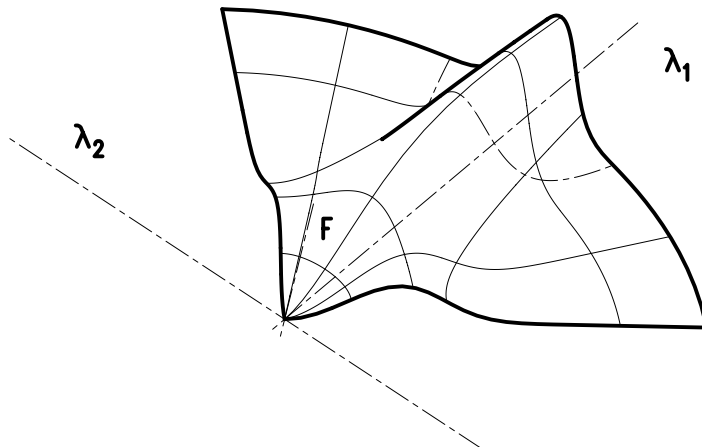


Figure 4: Fermionic part of the probability density

can expect that considering the complete base (27) the ground state density will be the same near the origin as in the picture and also will have the hills on the valleys of the potential which will be sharper and sharper when we follow the potential valley to large r . One can not conjecture in this case the behavior of the hills by this numerical method. Rather one has to use other methods [12] for the asymptotic behavior of the wavefunction at large r .

4 Numerical calculation II

Here we briefly review the body of paper [6] where a different approach to solving the eigenvalue problem of a Hamiltonian is given. The main idea is to choose a finite subset of a complete basis of the Hamiltonian Hilbert space and then find the combination which minimizes the lowest lying energy for instance. A lot of Hamiltonians have the form (24) which can be easily rewritten in terms of the harmonic oscillator

$$-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\zeta^2 x^2 \quad (36)$$

creation and annihilation operators a_+, a_- . This is true even in the case where the potential $V(x)$ is not quadratic. The complete base above is formed by eigenvectors of this linear harmonic oscillator which are constructed by creation operators acting on the vacuum. If we choose a finite subset of this basis, calculating Hamiltonian matrix elements is very easy.

In the harmonic oscillator Hamiltonian (36) which we used to define the basic creation and annihilation operators there is an arbitrary parameter ζ in the potential which we may use to optimize the numerics. The option to use ζ to

optimize the calculations was invented by us and should be regarded as suggestion for an improvement of the method presented in [6]. Concretely we will do it like this: first we choose a subset of the basis functions, we then compute the energy (or whatever else we would like to calculate) for different values of ζ . We then find the value of ζ which gives the minimum value for the energy. It turns out that it is much more effective to minimize the energy in terms of ζ than to try to use a larger basis since the computer time used for a calculation increases exponentially with the number of basis functions included.

There is one extra complication in our problem. Because of the gauge symmetry one should also restrict the basis to include only gauge invariant states. The generalization to the fermionic part and to the higher dimensional problem is straightforward.

Let us illustrate the method above on our Hamiltonian. First of all we have to form all gauge invariant states which can be obtained by acting with a combination of creation operators on the vacuum. The $SU(2)$ algebra has only two tensors which can form scalars from vectors

$$\delta^{ab}, \epsilon^{abc}. \quad (37)$$

These vectors are formed by creation operators a_{+i}^a, χ^a and a combination of them has to be contracted in gauge indexes with the special tensors above to get a gauge scalar. Let us write all non zero fundamental possibilities

$$\begin{aligned} & \delta^{ab} a_{+i}^a a_{+j}^b \\ & \delta^{ab} a_{+i}^a \chi^b \\ & \epsilon^{abc} a_{+i}^a a_{+j}^b \chi^c \\ & \epsilon^{abc} a_{+i}^a \chi^b \chi^c \\ & \epsilon^{abc} \chi^a \chi^b \chi^c \end{aligned} \quad (38)$$

where any combination of them acting on a gauge invariant state gives also a gauge invariant state. One has to start with the vacuum to generate all gauge invariant state. However, if we are looking for the groundstate we are interested only in states which can form a state with total spin zero. That is only states which consist of the fermion vacuum or two creation fermion operators acting on it because these are the only cases where the fermionic contribution to the total angular momentum can be canceled by the bosonic angular momentum. So using the operators

$$a_{+i}^a a_{+j}^a \quad i, j = 1, 2 \quad (39)$$

acting on $|0\rangle$ and on the fermion states

$$\begin{aligned} |1'\rangle &= \frac{1}{\sqrt{12}} a_{+1}^a \epsilon^{abc} \chi^b \chi^c |0\rangle \\ |2'\rangle &= \frac{1}{\sqrt{12}} a_{+2}^a \epsilon^{abc} \chi^b \chi^c |0\rangle \end{aligned} \quad (40)$$

gives the appropriate complete basis for our Hamiltonian. Choosing a finite subset of the basis states allows us to calculate matrix elements of any operator on the computer. It is a good idea to orthonormalize the states in the basis to avoid problems in the eigenvalue calculation when the metric is non trivial. See the next section for the explicit form of first few states in the basis.

We are able to illustrate the effect of the optimized ζ now. Let us show what happens with the ground state energy when we calculate with fixed ζ but with different coupling constants $g_s \in (0, 2]$ in our Hamiltonian (5). In this calculation we have chosen the restriction of the basis to include up to six creation operators in the bosonic basis and up to five creation operators in the fermion part. The result is given by the dotted line in the figure 5.

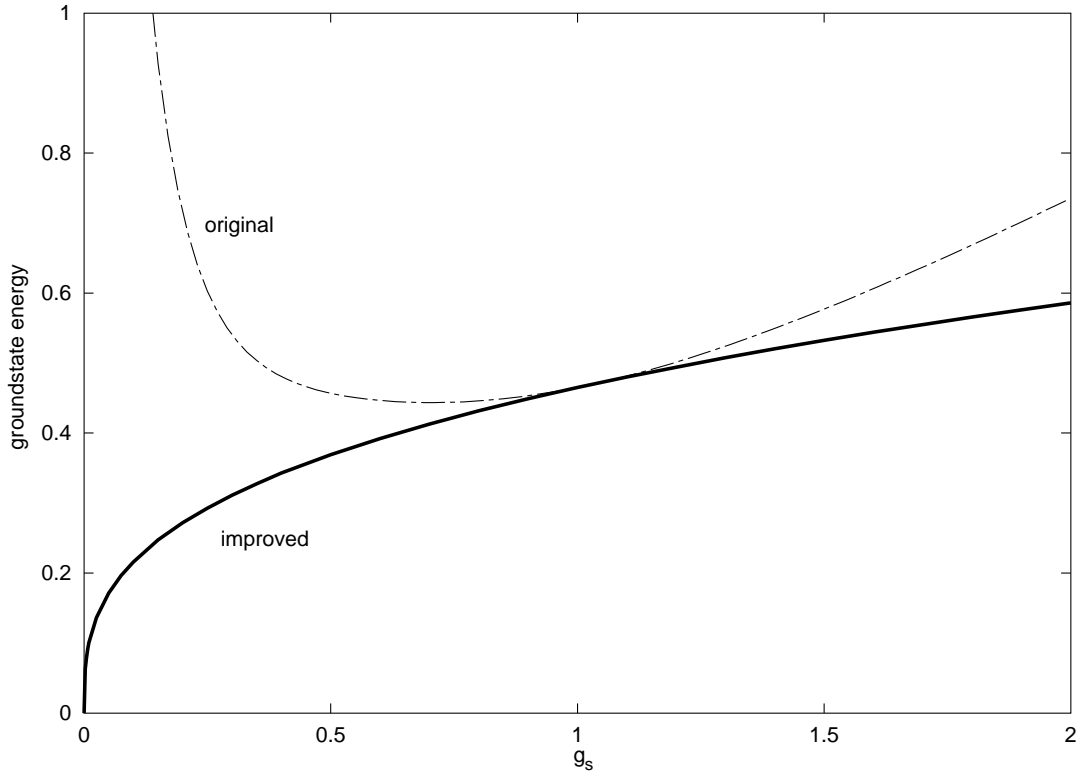


Figure 5: Groundstate energy dependence on the g_s

It seems that the energy of the ground state *diverges* when the coupling constant goes to zero which is clearly wrong! The way out of this dilemma is to remember that we also have the parameter ζ at our disposal. If we for each g_s use the ζ which gives the minimal energy, we get the dependence of the ground state energy on the coupling constant as drawn in bold line. This is clearly consistent with the $g_s^{1/3}$ dependence one expects from dimensional arguments. This is a very nice illustration of the power of our suggestion for improving the Wosiek

method [6] by including the ζ parameter. One can guess that there has to exist a coupling constant such that the optimization is not needed. This is also viewable from the picture.

There are also some results for coupling constant $g_s = 0.1$ in the (Tab. 1, p. 17) where the meaning of the columns is described above the table. The energy of the bold line for $g_s = 0.1$ corresponds to the first two columns in the table for the highest order obtained by the Wosiek method. We draw the density plot (Fig. 6.) for this calculation on which we can see the bump which for a larger number of basis functions develops into the long and thin hill in the picture on page 10. Its domain is $(r, \theta) \in [0, 1.5] \times [-\pi/4, \pi/4]$.

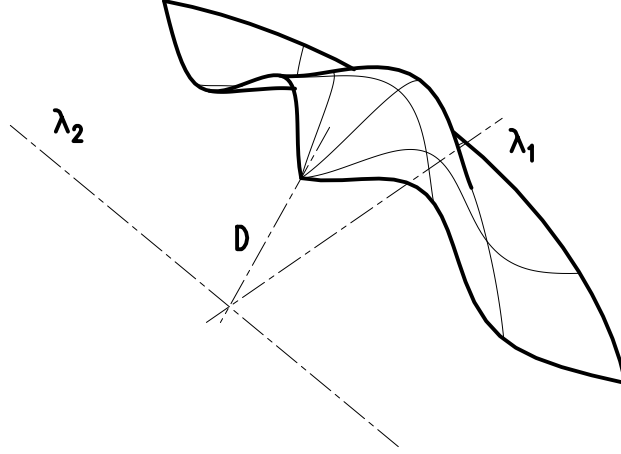


Figure 6: Groundstate by Wosiek method

5 Comparison

To compare our method with the Wosiek method it is important to know the correspondence between the base (33,34) and the base formed by creation operators. Since we have written our Hamiltonian in terms of angular momentum operators it is useful to classify all states in terms of angular momenta. In fact, this is even more useful since when we are discussing the ground state with total spin zero. We may rewrite the angular momentum operator (12) as

$$i(a_{+2}^a a_{-1}^a - a_{+1}^a a_{-2}^a) \quad (41)$$

using the correspondence between creation/annihilation and X_i^a, π_i^a operators

$$a_{\pm i}^a = \sqrt{\zeta/2}(X_i^a \mp i/\zeta \pi_i^a) . \quad (42)$$

Let us write down how the lowest lying angular momentum states look like. First we do the purely bosonic sector. The (zero angular momentum) ground state $|0\rangle$ in the coordinate representation looks like

$$\psi_0(r) = \left(\frac{\zeta}{\pi}\right)^{\frac{3}{2}} e^{-\frac{1}{2}\zeta r^2}. \quad (43)$$

Next we can write down all gauge invariant states created by two bosonic oscillators.

$$\begin{aligned} |1'\rangle &= \frac{1}{\sqrt{6}} a_{+1}^a a_{+1}^a |0\rangle \\ |2'\rangle &= \frac{1}{\sqrt{6}} a_{+2}^a a_{+2}^a |0\rangle \\ |3'\rangle &= \frac{1}{\sqrt{3}} a_{+1}^a a_{+2}^a |0\rangle . \end{aligned} \quad (44)$$

The following linear combinations

$$\begin{aligned} &\frac{i}{2} |1'\rangle - \frac{i}{2} |2'\rangle + \frac{1}{\sqrt{2}} |3'\rangle \\ &\quad \frac{1}{\sqrt{2}} |1'\rangle + \frac{1}{\sqrt{2}} |2'\rangle \\ &-\frac{i}{2} |1'\rangle + \frac{i}{2} |2'\rangle + \frac{1}{\sqrt{2}} |3'\rangle \end{aligned} \quad (45)$$

are orthonormal eigenvectors of the angular momentum operator with eigenvalues $\{-2, 0, 2\}$. Using formulas for the creation/annihilation operators in the X_i^a, π_i^a representation we may write the coordinate representation of these states as

$$\begin{aligned} &\frac{i}{\sqrt{6}} r^2 \zeta \cos 2\theta e^{-2i\phi} \psi_0 \\ &\quad \frac{1}{\sqrt{3}} (-3 + r^2 \zeta) \psi_0 \\ &-\frac{i}{\sqrt{6}} r^2 \zeta \cos 2\theta e^{2i\phi} \psi_0 . \end{aligned} \quad (46)$$

The angular momenta spectrum of the next six states all formed by four creation operators is $\{-4, -2, 0, 2, 4\} \oplus \{0\}$ where the two orthonormal states with zero angular momenta are given by

$$\begin{aligned} &\frac{1}{4\sqrt{105}} (60 - 40r^2\zeta + 5r^4\zeta^2 + 3r^4\zeta^2 \cos 4\theta) \psi_0 \\ &\quad \frac{1}{4\sqrt{21}} (36 - 24r^2\zeta + 3r^4\zeta^2 - r^4\zeta^2 \cos 4\theta) \psi_0 . \end{aligned} \quad (47)$$

They give the first non trivial wave function dependence on θ in the zero angular momentum sector. The θ part in higher order states with zero angular momenta reproduce the basis (33) which we guessed as the Fourier basis for the functions with the correct boundary conditions. We see that a new $\cos 4j\theta$ appears when there appears a new 0 in the angular momenta spectrum.

For reference we also give the spectra for the states created by six and eight creation operators $\{-6, -4, -2, 0, 2, 4, 6\} \oplus \{-2, 0, 2\}$ and $\{-8, -6, -4, -2, 0, 2, 4, 6, 8\} \oplus \{-4, -2, 0, 2, 4\} \oplus \{0\}$. With this we close the discussion about the purely bosonic part.

In the same way we can study the fermionic part. The wave function that interest us most are the ones with angular momentum one since they are the ones which contribute to the ground state with total angular momentum zero. On the lowest level in the creation/annihilation operator basis we have two gauge invariant orthonormal states

$$\begin{aligned} |1'\rangle &= \frac{1}{\sqrt{12}} a_{+1}^a \epsilon^{abc} \chi^b \chi^c |0\rangle \\ |2'\rangle &= \frac{1}{\sqrt{12}} a_{+2}^a \epsilon^{abc} \chi^b \chi^c |0\rangle . \end{aligned} \quad (48)$$

They can be combined into the orthonormal states

$$\begin{aligned} &\frac{i}{\sqrt{2}} |1'\rangle + \frac{1}{\sqrt{2}} |2'\rangle \\ &-\frac{i}{\sqrt{2}} |1'\rangle + \frac{1}{\sqrt{2}} |2'\rangle \end{aligned} \quad (49)$$

which are eigenstates of angular momenta with eigenvalues $\{-1, 1\}$. It is again possible to rewrite these states in the coordinate basis with help of the orthonormal and gauge invariant purely fermionic states (16) as

$$\begin{aligned} &\frac{i}{\sqrt{3}} r \sqrt{\zeta} \cos \theta e^{-i\phi} |1\rangle + \frac{1}{\sqrt{3}} r \sqrt{\zeta} \sin \theta e^{-i\phi} |2\rangle \\ &-\frac{i}{\sqrt{3}} r \sqrt{\zeta} \cos \theta e^{i\phi} |1\rangle + \frac{1}{\sqrt{3}} r \sqrt{\zeta} \sin \theta e^{i\phi} |2\rangle . \end{aligned} \quad (50)$$

The θ part of the second state gives the first function in the basis (34). The states in the next level consisting of the six states we get by acting with all gauge invariant combinations of two bosonic creation operators on the previous two states have angular momentum spectrum $\{-3, -1, 1, 3\} \oplus \{-1, 1\}$ where the two orthonormal states with angular momentum one are given by

$$\begin{aligned} &\frac{1}{\sqrt{18}} r \sqrt{\zeta} \cos \theta (2 - r^2 \zeta + r^2 \zeta \cos 2\theta) e^{i\phi} |1\rangle \\ &-\frac{i}{\sqrt{18}} r \sqrt{\zeta} \sin \theta (-2 + r^2 \zeta + r^2 \zeta \cos 2\theta) e^{i\phi} |2\rangle \end{aligned}$$

$$\begin{aligned}
& -\frac{i}{\sqrt{90}}r\sqrt{\zeta}\cos\theta(-10+2r^2\zeta+r^2\zeta\cos 2\theta)e^{i\phi}|1\rangle \\
& -\frac{1}{\sqrt{90}}r\sqrt{\zeta}\sin\theta(10-2r^2\zeta+r^2\zeta\cos 2\theta)e^{i\phi}|2\rangle .
\end{aligned} \tag{51}$$

This agrees with the second function in our fermionic basis. Continuing we find that the θ part of the angular momentum one states at higher levels reproduce the basis (34) which we have used. Writing the angular momentum spectrum for the next level (states obtained by acting with all gauge invariant combinations of four bosonic oscillators on the original fermionic states) we have $\{-5, -3, -1, 1, 3, 5\} \oplus \{-3, -1, 1, 3\} \oplus \{-1, 1\}$. This closes the discussion about the fermionic part and again gives a hint how the general spectrum looks like.

We have now achieved our main goal to rewrite the states in the Wosiek method in terms of angular momentum eigenstates in order to be able to compare the two methods. Since we now know at what level a particular basis function appears in the Wosiek method, we know where we should “cut off” the set of basis functions used in our method in order to get an equivalent calculation. Furthermore, doing the calculation using the two different methods at the same level we should still expect that our method should give better results since in our method the radial part of the wave function is not a priori fixed while in the Wosiek method the radial part comes together with the angular part, see for instance (47). Also, our method should be more effective for fixed spin calculation on the computer since we have the possibility to use only the angular momentum eigenstates we need while in the Wosiek method one is using all possible states which leads to a huge number of states not contributing to the wavefunction but absorbing space and time in the computer.

To illustrate this we compose the following table. We compare the (improved)

N_p	ζ	E_p	N	E
4+2	4.55	0.395	1+1	0.376
10+8	4.55	0.303	2+2	0.295
20+20	4.73	0.216	2+3	0.214
1540+440			10+10	0.075
11480+3080			20+20	0.037
37820+9920			30+30	0.025
			40+40	0.018
			50+50	0.014
			80+80	0.008

Table 1: Results

Wosiek method with N_p boson and fermion creation/annihilation states of the

type (44,48) with our method using only the test functions (33,34) to the corresponding level for $g_s = 0.1$. The ζ given in the table is the optimized value for the (improved) Wosiek method, the N counts the number of the test functions corresponding to the level used in the Wosiek method and the groundstate energies obtained by both methods are also given.

We see that indeed the energy calculated using our method is lower than in the Wosiek method. Furthermore it is clear that the number of states used in the Wosiek method increases uncontrollably as compared to the increase of basis functions used in our method.

Finally we also compare probability density sections for $\theta = 0$ in the case of the third line in the table (the last case where both methods have been used) on the domain $r \in [0, 2]$. The states which generate these sections have the same norm to compare them correctly. We have also included higher order wave

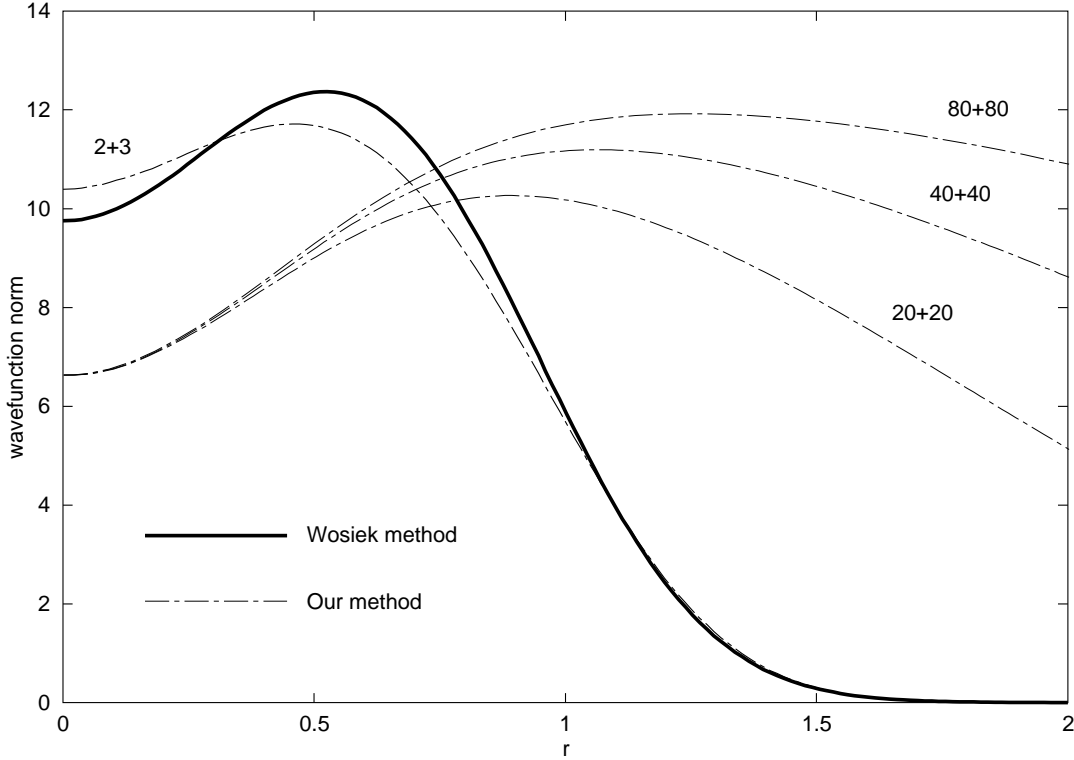


Figure 7: Comparison of wavefuctions at $\theta = 0$

functions calculated using our method. The labels indicate which order they correspond to. These wave functions are not equally normalized but have rather been chosen to have the same value at $r = 0$ in order to illustrate the behavior of the wavefunction when the order is increased. We see that the maximum moves outwards at the same time as the wavefunction becomes wider.

6 Conclusions

The aim of the present work was to find an approach how to calculate the bound-state of two 0-branes. This we have achieved with good results. In the course of our work we also found a way to improve the Wosiek method but our method still gives better results. As this work was being completed the closely related paper [17] by Wosiek (which improve his original method) appeared.

We have also calculated the groundstate probability density near the origin with high accuracy which gives a basic intuition about the physics of branes on the string scale. In particular about the meaning of the auxilliary coordinates which become important at small distances between the branes. It is for example interesting to observe that the most probable position of the branes is not on top of each other but rather at some small distance away from each other. This we understand as an effect of the Pauli fermionic repulsion.

It is possible to apply the method described here for branes in higher dimensional Minkowski space. However there are a lot of complications which do not allow numerical results of high precision. Namely, there are many more auxilliary coordinates in the higher dimensional case (or in the case where there are more than two branes). Since our basis functions are functions of the auxilliary coordinates, this implies that our basis functions will be multi dimensional functions and thus the number of components that the computer has to handle increases. Naively speaking the number of effective radial wavefunction in the state computed by the Numerov method for a given dimension is the number in our toy model raised to the power of the number of auxilliary coordinates. This fact of course taxes the computer very heavily. Another important point that we wanted to study was how the branes behave in external (generalized electric-magnetic) fields. The potential coming from these background fields represents this interaction. However, our two dimensional toy model is too simple to display any interesting effects. Again one would have to go to higher dimensional Minkowski space to do be able to do some interesting calculations.

As a first attempt to go further it would therefore be nice to consider three dimensional Minkowski space which would allow more general background fields which coming from other types of branes. However, we do feel that doing *realistic* calculations of the full ten dimensional (or even four dimensional) theory or including more than two 0-branes would not be possible even on a super computer. This is one of the reasons why we have not pursued this issue further.

Acknowledgments

I am very grateful to Rikard von Unge for continuous support and I also thank to Milan Šindelka which told me of his experiences in numerical methods which can solve quantum mechanics problems.

A Metric derivation

It is possible to decompose X_i^a [12] in matrix form as (6)

$$X = \psi \Lambda \eta \quad (\text{A.1})$$

where ψ is a group element in the adjoint representation of the gauge group and η is a group element of the rotation group $SO(2)$. The matrix Λ is then gauge and rotation invariant. We may now choose our physical coordinates as

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \\ 0 & 0 \end{pmatrix}, \quad \lambda_1 = r \cos \theta; \quad \lambda_2 = r \sin \theta \quad (\text{A.2})$$

together with

$$\eta = \exp(\phi T) , \quad (\text{A.3})$$

where T is the generator of spatial rotation

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \quad (\text{A.4})$$

The matrix ψ which describes the gauge degrees of freedom is parametrized by Euler angles α, β, γ

$$\psi = \exp(\alpha T_3) \exp(\gamma T_1) \exp(\beta T_3) \quad (\text{A.5})$$

where the matrices are generators of $SU(2)$ in the adjoint representation.

The flat metric in the original X_i^a coordinates is trivial and can be written in matrix notation as

$$g = \text{Tr}(dX^T dX) . \quad (\text{A.6})$$

The decomposition (6) leads to the expression

$$dX = d\psi \Lambda \eta + \psi d\Lambda \eta + \psi \Lambda d\eta \quad (\text{A.7})$$

and then the metric is

$$\text{Tr}(d\Lambda^T d\Lambda - \psi^T d\psi \Lambda \Lambda^T \psi^T d\psi + d\eta^T \Lambda^T \Lambda d\eta + 2\eta d\eta^T \Lambda^T \psi^T d\psi \Lambda) . \quad (\text{A.8})$$

We see here that the differentials in $d\Lambda$ are not mixed with the other ones and thus the metric will be block diagonal

$$\begin{pmatrix} g_i & \\ & g_m \end{pmatrix} \quad (\text{A.9})$$

where the g_i block is associated with the $d\Lambda$ differential and g_m is associated with the differentials $d\eta, d\psi$. Now we compute the metric term by term. The Λ part in the case of two branes is

$$\text{Tr}(d\Lambda^T d\Lambda) = d\lambda_i d\lambda_i \quad (\text{A.10})$$

or explicitly in variables r, θ from (A.2)

$$d\lambda_i d\lambda_i = dr^2 + r^2 d\theta^2. \quad (\text{A.11})$$

Next we take care of the terms which contribute to the g_m part of the metric. Using the fact that

$$\psi^{-1} = \psi^T \quad (\text{A.12})$$

the purely gauge part

$$- \text{Tr}(\psi^T d\psi \Lambda \Lambda^T \psi^T d\psi) \quad (\text{A.13})$$

can be written with help of the left-invariant one forms defined by the ψ being a group element in the adjoint representation of $SU(2)$

$$\omega_L = \psi^{-1} d\psi = \omega_{La} T_a \quad (\text{A.14})$$

where T_a are the generators. In our coordinates we explicitly have the expressions

$$\begin{aligned} \omega_{L1} &= \sin \gamma \sin \beta d\alpha + \cos \beta d\gamma \\ \omega_{L2} &= -\sin \gamma \cos \beta d\alpha + \sin \beta d\gamma \\ \omega_{L3} &= \cos \gamma d\alpha + d\beta. \end{aligned} \quad (\text{A.15})$$

Introducing the matrix Ω_L of the coefficient of the ω 1-forms

$$\begin{pmatrix} \omega_{L1} \\ \omega_{L2} \\ \omega_{L3} \end{pmatrix} = \underbrace{\begin{pmatrix} \sin \gamma \sin \beta & 0 & \cos \beta \\ -\sin \gamma \cos \beta & 0 & \sin \beta \\ \cos \gamma & 1 & 0 \end{pmatrix}}_{\Omega_L} \begin{pmatrix} d\alpha \\ d\beta \\ d\gamma \end{pmatrix} \quad (\text{A.16})$$

the gauge part (A.13) is

$$e^T \Omega_L^T \Pi \Omega_L e \quad (\text{A.17})$$

where $e^T = (d\alpha, d\beta, d\gamma)$ and the diagonal matrix $\Pi_{ab} = \text{Tr}(T_a \Lambda \Lambda^T T_b)$ has elements

$$\Pi_{aa} = \sum_{i \neq a} \lambda_i^2. \quad (\text{A.18})$$

The calculation of the contribution from terms which contain $d\eta$ is very similar. Introducing the right-invariant one form defined by $d\eta\eta^{-1}$ and using our parametrisation (A.3) we have

$$\text{Tr}(d\eta^T \Lambda^T \Lambda d\eta) = r^2 d\phi^2 \quad (\text{A.19})$$

and

$$\text{Tr}(2\eta d\eta^T \Lambda^T \psi^T d\psi \Lambda) = 2r^2 \sin 2\theta d\phi \omega_{L3} . \quad (\text{A.20})$$

In this expression it is necessary to write the ω_L (A.15) explicitly. Since we know that the metric is block diagonal (A.9) we will write the g_m part of the metric compactly. This is done by writing the matrix (15)

$$\mathbf{\Pi} = \begin{pmatrix} r^2 & & & r^2 \sin 2\theta \\ & r^2 \sin^2 \theta & & \\ & & r^2 \cos^2 \theta & \\ r^2 \sin 2\theta & & & r^2 \end{pmatrix} \quad (\text{A.21})$$

and the structure matrix

$$\mathbf{\Omega}_L = \begin{pmatrix} 1 & \\ & \Omega_L \end{pmatrix} . \quad (\text{A.22})$$

Then it is possible to write the sum of (A.17, A.19, A.20) as

$$\mathbf{e}^T \mathbf{\Omega}_L^T \mathbf{\Pi} \mathbf{\Omega}_L \mathbf{e} \quad (\text{A.23})$$

where $\mathbf{e}^T = (d\phi, d\alpha, d\beta, d\gamma)$. So

$$g_m = \mathbf{\Omega}_L^T \mathbf{\Pi} \mathbf{\Omega}_L \quad (\text{A.24})$$

and the full metric is

$$dr^2 + r^2 d\theta^2 + \mathbf{e}^T \mathbf{\Omega}_L^T \mathbf{\Pi} \mathbf{\Omega}_L \mathbf{e} . \quad (\text{A.25})$$

It is now easy to find the square root of the metric determinant

$$\frac{1}{4} r^5 \sin \gamma \sin 4\theta \quad (\text{A.26})$$

which also gives us the integration measure. On the domain of $(r, \theta, \phi, \alpha, \beta, \gamma)$

$$(0, \infty) \times [0, \pi/4] \times (0, 2\pi] \times (0, 2\pi] \times (0, 2\pi] \times (0, \pi] \quad (\text{A.27})$$

the transformation rule (6) is one to one and on its inside the Jacobian is nonzero and thus regular which is important in calculation of derivatives with respect to X_i^a in terms of these coordinates.

B Laplacian derivation

The metric in the new coordinates is (A.25)

$$dr^2 + r^2 d\theta^2 + \mathbf{e}^T \boldsymbol{\Omega}_L^T \boldsymbol{\Pi} \boldsymbol{\Omega}_L \mathbf{e} \quad (\text{B.1})$$

where $\mathbf{e}^T = (d\phi, d\alpha, d\beta, d\gamma)$, $\boldsymbol{\Omega}_L$ is defined by (A.22) and $\boldsymbol{\Pi}$ by (15). From the metric or directly from (A.8) one can see that the metric has the block diagonal structure (A.9)

$$\begin{pmatrix} g_i(r) & \\ & g_m(r, \theta, \alpha, \beta, \gamma) \end{pmatrix} \quad (\text{B.2})$$

where g_i is a diagonal matrix being the metric in the r, θ variables and g_m is the metric in the other variables. Thus we are able to calculate the inverse of the full metric in terms of the inverses of g_i and g_m separately. Since g_i does not depend on the gauge variables, the Laplacian also splits into two pieces

$$\frac{1}{\sqrt{\det g_i \det g_m}} \partial_j (\sqrt{\det g_i \det g_m} g_i^{jj} \partial_j) + \frac{1}{\sqrt{\det g_m}} \partial_\mu (\sqrt{\det g_m} g_m^{\mu\nu} \partial_\nu) \quad (\text{B.3})$$

where the latin index runs over the $\{r, \theta\}$ variables and the greek indexes over the angular variables. Using the square root of the metric determinant (A.26) the first part of the Laplacian generated by g_i is

$$\frac{1}{\sqrt{\det g}} \partial_j (\sqrt{\det g} g_i^{jj} \partial_j) = \frac{1}{r^5} \frac{\partial}{\partial r} r^5 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin 4\theta} \frac{\partial}{\partial \theta} \sin 4\theta \frac{\partial}{\partial \theta} . \quad (\text{B.4})$$

At this point we would like to rewrite the second part of the Laplacian in terms of the angular momenta (12,11)

$$\begin{aligned} L^0 &= -i \frac{\partial}{\partial \phi} \\ L^1 &= -i \cot \gamma \sin \alpha \frac{\partial}{\partial \alpha} + i \csc \gamma \sin \alpha \frac{\partial}{\partial \beta} + i \cos \alpha \frac{\partial}{\partial \gamma} \\ L^2 &= -i \cot \gamma \cos \alpha \frac{\partial}{\partial \alpha} + i \csc \gamma \cos \alpha \frac{\partial}{\partial \beta} - i \sin \alpha \frac{\partial}{\partial \gamma} \\ L^3 &= i \frac{\partial}{\partial \alpha} . \end{aligned} \quad (\text{B.5})$$

We notice that the angular momenta are equal to the vector field X_R being dual to the *right* invariant one form $\omega_R = d\psi\psi^{-1}$

$$\begin{aligned} -iL_1 = X_{R1} &\equiv -\cot \gamma \sin \alpha \frac{\partial}{\partial \alpha} + \csc \gamma \sin \alpha \frac{\partial}{\partial \beta} + \cos \alpha \frac{\partial}{\partial \gamma} \\ -iL_2 = X_{R2} &\equiv -\cot \gamma \cos \alpha \frac{\partial}{\partial \alpha} + \csc \gamma \cos \alpha \frac{\partial}{\partial \beta} - \sin \alpha \frac{\partial}{\partial \gamma} \\ -iL_3 = X_{R3} &\equiv \frac{\partial}{\partial \alpha} . \end{aligned} \quad (\text{B.6})$$

Since the right invariant one forms induce left translations, they are Killing fields of the metric. Also, since we know the relation between the left and right invariant one forms

$$\omega_R = d\psi\psi^{-1} = \psi\omega_L\psi^{-1} \quad (\text{B.7})$$

we may transform the expression for the metric, which is expressed in terms of X_L into an expression in terms of X_R which is then equivalent to an expression in terms of the angular momenta. Explicitly we write the vector field $X_{L,R}$ in terms of coefficient matrices $\Theta_{L,R}$ inverse to the matrices $\Omega_{L,R}$ introduced in (A.16). The relation (B.7) then implies

$$\Theta_L = \psi^{-1}\Theta_R. \quad (\text{B.8})$$

To treat all the angles on equal footing, we incorporate also the physical angle ϕ into this formalism by extending the three by three matrices Ω_R, Θ_R, ψ to four by four matrices by putting -1 in the "zeroth" row and column. These new matrices we denote by bold letters, for instance

$$\mathbf{\Psi} = \begin{pmatrix} -1 & \\ & \psi \end{pmatrix}. \quad (\text{B.9})$$

Now all vector fields obtained from Θ_R are equal to $-i$ times the physical and the gauge angular momenta and they are all Killing vectors of the metric. We can now write $\mathbf{\Omega}_L = \mathbf{\Psi}^{-1}\mathbf{\Omega}_R$; $\mathbf{\Theta}_L = \mathbf{\Psi}^{-1}\mathbf{\Theta}_R$.

We may now rewrite the angular part of the Laplacian in terms of the angular momenta which are given by $\mathbf{\Theta}_R$. The appropriate part in the Laplacian is

$$\begin{aligned} & \mathbf{\Theta}_L^T \mathbf{\Pi}^{-1} \mathbf{\Theta}_L \\ &= \mathbf{\Theta}_R^T \mathbf{\Psi} \mathbf{\Pi}^{-1} \mathbf{\Psi}^{-1} \mathbf{\Theta}_R. \end{aligned} \quad (\text{B.10})$$

First we expand the formula for the Laplacian

$$\frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu) = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g}) g^{\mu\nu} \partial_\nu + \partial_\mu (g^{\mu\nu}) \partial_\nu + g^{\mu\nu} \partial_\mu \partial_\nu. \quad (\text{B.11})$$

The first term is reduced to

$$- \frac{\partial}{\partial x^\xi} (\mathbf{\Theta}_R)_\sigma^\xi (\mathbf{\Psi} \mathbf{\Pi}^{-1} \mathbf{\Psi}^{-1} \mathbf{\Theta}_R)_\sigma^\rho \frac{\partial}{\partial x^\rho} \quad (\text{B.12})$$

where we have used the Killing equation for $\mathbf{\Theta}_R$. In the same way the second term becomes

$$\frac{\partial}{\partial x^\xi} (\mathbf{\Theta}_R)_\sigma^\xi (\mathbf{\Psi} \mathbf{\Pi}^{-1} \mathbf{\Psi}^{-1} \mathbf{\Theta}_R)_\sigma^\rho \frac{\partial}{\partial x^\rho} + (\mathbf{\Theta}_R)_\sigma^\xi (\mathbf{\Psi} \mathbf{\Pi}^{-1} \mathbf{\Psi}^{-1})_{\sigma\zeta} \frac{\partial}{\partial x^\xi} (\mathbf{\Theta}_R)_\zeta^\rho \frac{\partial}{\partial x^\rho} \quad (\text{B.13})$$

and finally the third part is

$$(\Theta_R)_\sigma^\xi (\Psi \Pi^{-1} \Psi^{-1})_{\sigma\zeta} (\Theta_R)_\zeta^\rho \frac{\partial}{\partial x^\xi} \frac{\partial}{\partial x^\rho} . \quad (\text{B.14})$$

So the sum of these three contributions gives

$$\begin{aligned} & (\Theta_R)_\sigma^\xi (\Psi \Pi^{-1} \Psi^{-1})_{\sigma\zeta} (\Theta_R^T)_\zeta^\rho \frac{\partial}{\partial x^\xi} \frac{\partial}{\partial x^\rho} + (\Theta_R)_\sigma^\xi (\Psi \Pi^{-1} \Psi^{-1})_{\sigma\zeta} \frac{\partial}{\partial x^\xi} (\Theta_R)_\zeta^\rho \frac{\partial}{\partial x^\rho} \\ &= (\Psi \Pi^{-1} \Psi^{-1})_{\sigma\zeta} (\Theta_R)_\sigma^\xi \frac{\partial}{\partial x^\xi} \left[(\Theta_R)_\zeta^\rho \frac{\partial}{\partial x^\rho} \right] = -(\Psi \Pi^{-1} \Psi^{-1})_{\mu\nu} L^\mu L^\nu \end{aligned} \quad (\text{B.15})$$

where we have used that the angular momenta are defined in terms of the matrix Θ_R . Then the whole Laplacian is

$$\frac{1}{r^5} \frac{\partial}{\partial r} r^5 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin 4\theta} \frac{\partial}{\partial \theta} \sin 4\theta \frac{\partial}{\partial \theta} - (\Psi \Pi^{-1} \Psi^{-1})_{\mu\nu} L^\mu L^\nu . \quad (\text{B.16})$$

References

- [1] D. Bigatti and L. Susskind, “Review of matrix theory,” [arXiv:hep-th/9712072].
- [2] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, “M theory as a matrix model: A conjecture,” Phys. Rev. D **55** (1997) 5112 [arXiv:hep-th/9610043].
- [3] P. Hořava, “Type IIA D-branes, K-theory, and matrix theory,” Adv. Theor. Math. Phys. **2** (1999) 1373 [arXiv:hep-th/9812135].
- [4] E. Witten, “D-branes and K-theory,” JHEP **9812** (1998) 019 [arXiv:hep-th/9810188].
- [5] R. C. Myers, “Dielectric-branes,” JHEP **9912** (1999) 022 [arXiv:hep-th/9910053].
- [6] J. Wosiek, “Spectra of supersymmetric Yang-Mills quantum mechanics,” Nucl. Phys. B **644** (2002) 85 [arXiv:hep-th/0203116].
- [7] D. Kabat and P. Pouliot, “A Comment on Zero-brane Quantum Mechanics,” Phys. Rev. Lett. **77** (1996) 1004 [arXiv:hep-th/9603127].
- [8] M. Trzetrzelewski and J. Wosiek, “Quantum systems in a cut Fock space,” [arXiv:hep-th/0308007].

- [9] M. Campostrini and J. Wosiek, “Exact Witten index in $D = 2$ supersymmetric Yang-Mills quantum mechanics,” *Phys. Lett. B* **550** (2002) 121 [arXiv:hep-th/0209140].
- [10] J. Wosiek, “Supersymmetric Yang-Mills quantum mechanics,” [arXiv:hep-th/0204243].
- [11] B. de Wit, M. Luscher and H. Nicolai, “The Supermembrane Is Unstable,” *Nucl. Phys. B* **320** (1989) 135.
- [12] M. B. Halpern and C. Schwartz, “Asymptotic search for ground states of $SU(2)$ matrix theory,” *Int. J. Mod. Phys. A* **13** (1998) 4367 [arXiv:hep-th/9712133].
- [13] U. H. Danielsson, G. Ferretti and B. Sundborg, “D-particle Dynamics and Bound States,” *Int. J. Mod. Phys. A* **11** (1996) 5463 [arXiv:hep-th/9603081].
- [14] V. G. Kac and A. V. Smilga, “Normalized vacuum states in $N = 4$ supersymmetric Yang-Mills quantum mechanics with any gauge group,” *Nucl. Phys. B* **571** (2000) 515 [arXiv:hep-th/9908096].
- [15] E. Witten, “Bound states of strings and p-branes,” *Nucl. Phys. B* **460** (1996) 335 [arXiv:hep-th/9510135].
- [16] B. R. Johnson, “The renormalized Numerov method applied to calculating bound state of the coupled-channel Schroedinger equation,” *J. Chem. Phys.* **69** (1978) 10
- [17] J. Wosiek, “Recent progress in supersymmetric Yang-Mills quantum mechanics in various dimensions,” *Acta Phys. Polon. B* **34** (2003) 5103 [arXiv:hep-th/0309174].